TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 7

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Homework 7.1 (A general version of Bloch's theorem)**.** Show if $f\colon U \to \mathbf{C}$ is holomorphic and $f'(c) \neq 0$ at a point $c \in U$, then $f(U)$ contains disks of every radius $(3/2-\sqrt{2})s|f'(c)|$, where $s \in (0, \operatorname{dist}(c, \partial U))$. In particular, show if $f : \mathbf{C} \to \mathbf{C}$ is entire and nonconstant, then $f(\mathbf{C})$ contains disks of arbitrarily large radii.

Homework 7.2 (Biholomorphic functions on the unit ball)**.** Let $z_0 \in B_1(0)$ and define the function $\varphi_{z_0}\colon B_1(0) \to \mathbf{C}$ by

$$\varphi_{z_0}(z) := \frac{z - z_0}{1 - \overline{z_0}\, z}.$$

a. Show φ_{z_0} is a biholomorphic[1] map from $B_1(0)$ to $B_1(0)$.
b. Show every biholomorphic map $f\colon B_1(0) \to B_1(0)$ is of the form $f = a\,\varphi_{z_0}$, where $a \in \partial B_1(0)$[2].

Homework 7.3 (Improvement of the Bloch constant)**.** The purpose of this exercise is to improve the constant $3/2 - \sqrt{2}$ appearing in Bloch's theorem[3]. We will show that for every $f \in \mathcal{H}(\bar{B}_1(0))$ with $f'(0) = 1$ the image $f(B_1(0))$ contains a disk of radius $3\sqrt{2}/2 - 2$.

a. Motivated by Homework 7.1, we look for a function $F \in \mathcal{H}(\bar{B}_1(0))$ such that $f(B_1(0)) = F(B_1(0))$ with maximal value $|F'(0)|$. To make this precise, set
$$\mathcal{F} := \{f \circ (a\,\varphi_{z_0}) : z_0 \in B_1(0),\ a \in \partial B_1(0)\},$$
where φ_{z_0} is from Homework 7.2. Show that every $h \in \mathcal{F}$ with $h = f \circ (a\,\varphi_{z_0})$ obeys $h \in \mathcal{H}^1(\bar{B}_1(0))$, $h(B_1(0)) = f(B_1(0))$ and $|h'(0)| = |f'(-az_0)|(1 - |z_0|^2)$.
b. Let q denote the maximizer of the assignment $|f'(z)|(1 - |z|^2)$, where $z \in \bar{B}_1(0)$. Show $q \in B_1(0)$ and that, setting $F := f \circ (\varphi_{-q})$, we have
$$|F'(z)| \leq \frac{|f'(q)|(1 - |q|^2)}{1 - |z|^2}.$$
c. Deduce $|F'(z)| \leq 2|F'(0)|$ whenever $|z| \leq \sqrt{2}/2$. Conclude the proof using Step 2 of the proof of Bloch's theorem in the lecture.

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[1] Recall a map is biholomorphic if it is bijective, holomorphic, and its inverse is holomorphic.

[2] **Hint.** Recall the Schwarz lemma.

[3] The optimal constant (say B) in Bloch's theorem is not known. There are lower and upper bounds. The best known thus far are

$$0.4332127\ldots = \frac{\sqrt{3}}{4} + 2 \cdot 10^{-4} \leq B \leq \frac{1}{\sqrt{1+\sqrt{3}}}\,\frac{\Gamma(1/3)\,\Gamma(11/12)}{\Gamma(1/4)} = 0.4718617\ldots$$

Homework 7.4 (Optional and difficult exercise)**.** Let $G \subset \mathbf{C}$ be a simply connected domain and let $f : G \to \mathbf{C}$ be holomorphic. We define $P := f(G) \cap \{-1, 1\}$. Show there exists a holomorphic function $g \colon G \to \mathbf{C}$ such that $f = \cos(g)$ if and only if for each $z_0 \in P$ the function $f - f(z_0)$ has a zero of even order[4].

[4]**Hint.** Show one can define the assignment $g(z) := -\mathrm{i}\log(f(z) + \sqrt{f(z)+1}\sqrt{f(z)-1})$. To define the square root, the Weierstraß product theorem might help.